The Open University
The Open University in the South East

MST121

REVISION

BOOKLET

MST121 Block A

Important ideas

- A sequence is an ordered set of terms.
- A recurrence relation is an equation relating each term in a sequence to previous term(s).
- The effect of varying parameters on the behaviour of
  - arithmetic sequences
  - geometric sequences
  - linear recurrence relations.
- Recurrence relations may be used to model many situations eg. populations, loan interest.
- Parametric representations.
- Discrete and continuous representations.
- Properties of a circle.
- Relative motion.
- Functions as rules and processes, inverse functions and the associated notation.
- Mathematical models are created to investigate "real world" problems.
- Models of apparently different situations can give rise to the same mathematical core.
- Independent learning - how you learn.

Important techniques

- Specify sequences by recurrence relations and closed forms.
- Conjecture a general result from special cases.
- Apply general results in specific instances.
- Find the distance between two points.
- Find the equation of a straight line, given sufficient information.
- Given the equation of a straight line, derive the slope, and intercepts on coordinate axes.
- Complete the square.
- Find the equation of a circle, given the radius and centre.
- Given the equation of a circle, derive the radius and centre.
- Recognise standard functions and their graphs.
- Find inverse functions.
- Use the modelling cycle to create simple mathematical models.
- Identify the simplifying assumptions made in a mathematical model.
- Use a computer (Mathcad) to help do/understand mathematics.

MST 121 Block A Contents

MST121 BLOCK A Problems

1. Solve

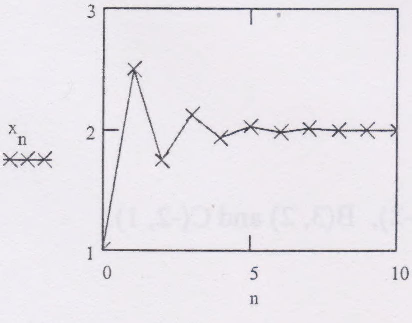
   (i)
$$\frac{2}{x-3} - \frac{3}{x+2} = 0$$

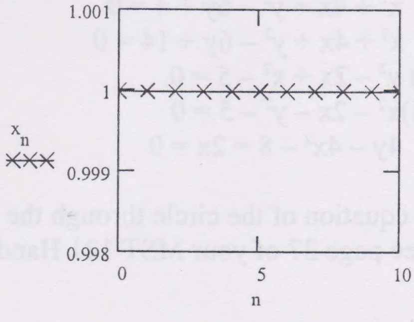
   (ii)
$$2x - 3y = 7$$
$$3x + 5y = 1$$

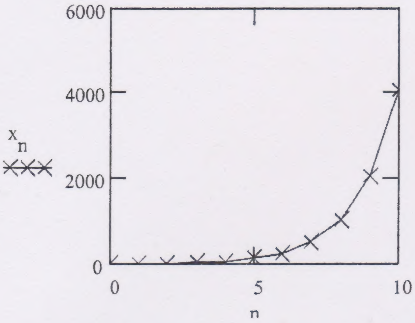
2. (a) For the following equations decide, giving your reasons, whether there are 0, 1 or 2 real solutions.
   (b) Solve those which have 1 or 2 solutions by factorisation **and** by using the formula.
   (i) $x^2 + 6x + 9 = 0$
   (ii) $2x^2 + x - 3 = 0$
   (iii) $x^2 - 9 = 0$
   (iv) $x^2 + 9 = 0$

3. Each of the recurrence systems below generates a sequence. For each sequence
   (a) write down the first 4 terms
   (b) write down the closed form
   (c) use part (b) to find the first 4 terms and confirm your answers to part (a)
   (d) find the 10th term.
   (i) $x_1 = 4000, \quad x_{n+1} = 0.8x_n \quad (n = 1,2,3\ldots)$
   (ii) $x_1 = -350, \quad x_{n+1} = x_n - 50 \quad (n = 1,2,3\ldots)$
   (iii) $x_0 = 1024, \quad x_{n+1} = -0.5x_n \quad (n = 0,1,2\ldots)$

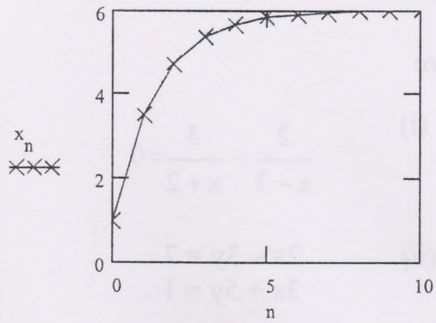
4. (a) For each of the following linear recurrence systems write down the first 4 terms, find the closed form of the sequence and describe its long term behaviour.
   (i) $x_0 = 1, \quad x_{n+1} = 0.5x_n + 3 \quad (n = 0,1,2\ldots)$
   (ii) $x_0 = 1, \quad x_{n+1} = 2x_n + 3 \quad (n = 0,1,2\ldots)$
   (iii) $x_0 = 1, \quad x_{n+1} = -0.5x_n + 3 \quad (n = 0,1,2\ldots)$
   (iv) $x_0 = 1, \quad x_{n+1} = -2x_n + 3 \quad (n = 0,1,2\ldots)$
   (b) The following graphs have been drawn by Mathcad. They represent the four linear recurrence systems but not in the given order. Decide which graph corresponds to which system.







(i)



(ii)

(iii) (iv)

5. x_n and y_n are infinite linear sequences. The first 4 terms of each sequence are given.

x_n	1, 3.5, 7.25, 12.875
y_n	20, -16, 9.2, -8.44

(a) Find the parameters for the corresponding recurrence systems and hence write down the recurrence systems, labelling the first terms as x_1 and y_1.
(b) Find the closed form for the sequences.
(c) Describe the long term behaviour of the sequences.

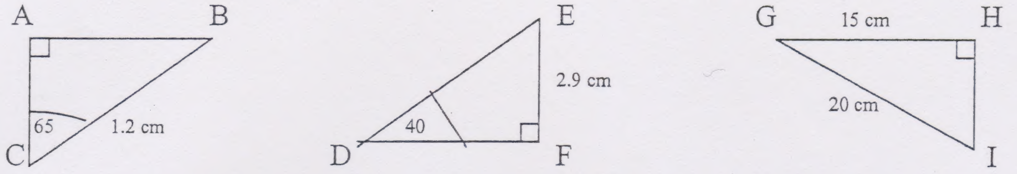
6. (a) Find
   (i) the distance between the points A(-2,-1) and B(2,2)
   (ii) the equation of the line through A and B.
(b) The parametric form of another line is $(t, 14/3 - 4t/3)$. Show that this line
   (i) is perpendicular to the line in (a), (ii) passes through B.
(c) Find the point of intersection of the two lines
   $y = 1/3\,(2x - 7)$ and $y = 1/5\,(1 - 3x)$
(d) Decide whether the following pairs of lines are parallel, perpendicular or neither.
   (i) $y = 3x + 5$, $y = 3x - 2$ (ii) $y = 3x + 5$, $y = 2 - 3x$ (iii) $y = 3x + 5$, $y = 2 - x/3$

7. Decide which of the following equations **do not** represent circles and why. For each which **does** represent a circle find its (a) centre, (b) radius, (c) parametric equations.
   (i) $(x - 2)^2 + (y + 3)^2 = 7$
   (ii) $(x - 2)^2 - (y + 3)^2 = 7$
   (iii) $(x - 2)^2 + 2(y + 3)^2 = 7$
   (iv) $(x - 2)^2 + (y + 3)^2 + 7 = 0$
   (v) $x^2 + 4x + y^2 - 6y + 4 = 0$
   (vi) $x^2 + 4x + y^2 - 6y + 14 = 0$
   (vii) $y^2 - 2x + x^2 - 5 = 0$
   (viii) $x^2 - 2x - y^2 - 5 = 0$
   (ix) $4y - 4x^2 - 8 = 2x = 0$

8. Find the equation of the circle through the points A(1, -2), B(3, 2) and C(-2, 1).
(Hint: see page 27 of your MST 121 Handbook)

9. Solve the following triangles.

(i) (ii) (iii)



10. For each of the quadratic functions given below
   (a) Rearrange each expression in completed square form;
   (b) Sketch the graph of the function using translations and scalings of the graph of $y = x^2$, finding the y-intercept and the x-intercepts if any.
   (i) $f(x) = x^2 + 6x + 17$
   (ii) $g(x) = 2x^2 - 8x + 3$
   (iii) $h(x) = 14x - 13 - x^2$

For the following questions you are reminded of the Domain Convention (see Handbook page 28).

11. Sketch the graphs of the following real functions using translations and scalings of the given graphs, and by finding the y-intercept and the x-intercepts if any.

(i) $f(x)=\dfrac{4}{(2x-5)} - 3$ from the graph of $y = 1/x$,

(ii) $g(x)=5-2\left|x-6\right|$ from the graph of $y = |x|$,

(iii) $h(x)=8\sin\left(2x-\dfrac{\pi}{4}\right)$ from the graph of $y = \sin(x)$

12. Solve the following equations:
   (a) (i) $5^x = 3125$ (ii) $5^x = 0.0016$ both without using your calculator
   (b) (i) $7^x = 3125$ (ii) $e^{-2x} = 37$
   (c) $\ln(x/3) = 17$

13. For each of the real functions given below
   (a) Sketch a graph of the function;
   (b) Decide whether the function has an inverse. If so, find it, stating its domain and image set.
      (i) $f(x) = 3x - 7$
      (ii) $g(x)=\dfrac{4}{2x-5} - 3$ see 11(i) above.
      (iii) $h(x) = 2x^2 - 8x + 3$ see 10(ii) above.

14. Find the inverses of the following functions, stating the domains and image sets
   (i) $f(x) = \cos(2x)$ (x in $[0, \pi/2]$)
   (ii) $g(x) = \tan(x/4)$ ($-2\pi < x < 2\pi$)
   (iii) $h(x) = 4\exp(-3x) + 7$
   (iv) $k(x) = 5\ln(3x - 2) - 8$

MST121 Block B

Important Ideas

- Mathematical modelling is the process of construction of a mathematical representation of a real situation.
- Mathematical models usually start from a simple representation of the real scenario and develop to a more complex model.
- Sequences have various forms of long term behaviour. Some sequences may not converge to a limit.
- It is always possible to find the number of terms in a converging sequence to place it within some stated tolerance of the limit of the sequence.
- The sums of a sequence form a series.
- A matrix is a rectangular array of numbers which may be used to hold quantitative information in a structured way.
- Vectors can be represented in different forms and which form is most appropriate depends on the modelling context.
- Forces can be modelled by using vectors.

Important Techniques

- Use the sigma notation to develop the closed form of a sequence.
- Construct discrete models for populations.
- Determine the long-term behaviour of a sequence.
- Evaluate the limit of a sequence.
- Develop a network and construct its associated matrix.
- Describe a matrix in terms of the numbers in its rows and columns.
- Evaluate matrix sums and products.
- Solve a simultaneous equation by matrix methods.
- Determine the inverse of a matrix if it exists.
- Convert vectors from column to geometric form.
- Convert vectors from geometric form to column form.
- Add, subtract and scalar multiply vectors.
- Use the Sine and Cosine Rules to solve triangles.
- Apply the Equilibrium Condition to find information about force vectors.
- Sketch force diagrams.
- Use Cartesian Unit vectors to simplify the component forms.
- Use Mathcad to assist with the modelling process.
- Use vectors to model displacements, velocities, acceleration and forces.

MST121 Block B Problems

1. Three cities, Paris, London and Rome have been designated as storage centres for Agricultural Aid distribution to Afghanistan. Based on previous years, the aid is distributed as shown in table B1. For example 30% of the aid from London arrives in Herat.

City / **Destination**	Paris	London	Rome
Kabul	30%	55%	15%
Herat	25%	30%	0%
Quandahar	45%	15%	85%

Table B1

(a) Draw a network diagram modelling the distribution from each city to the towns in Afghanistan.
(b) Form a matrix model which will be able to calculate the amount of Agricultural aid distributed to each town in Afghanistan.
(c) It has been decided to send seeds to Afghanistan from each centre as shown in table B2.

City	Paris	London	Rome
Metric tonne	1200	1500	2000

Table B2

Using the matrix model formed in (b), calculate the amount of seeds expected to arrive in the towns Kabul, Herat and Quandahar.

2. If $A := \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ and $B := \begin{pmatrix} -1 & 2 \\ 4 & 3 \end{pmatrix}$

(a) Evaluate **A+B**, **A-B**, **A**-2**B**, 3**A** – 4**B**, det (**A**) and det (**B**)
(b) Determine the inverse of **A**
(c) Determine the inverse of **B**
(d) Evaluate **AB** and **BA**. Does **AB** = **BA**?

3. Place, in the matrix format $A\begin{pmatrix} x \\ y \end{pmatrix} = b$ the simultaneous equations,

$$2x + 4y = 10$$
$$3x - 2y = -1$$

(a) By finding the determinant of **A,** write down a reason why this system of equations has a solution.
(b) Evaluate the inverse of **A** and hence calculate the solution vector.

4. If $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$ $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j}$ evaluate
   (a) $\mathbf{a} + \mathbf{b}$,
   (b) $\mathbf{a} - \mathbf{b}$,
   (c) $2\,\mathbf{a} - 3\,\mathbf{b}$,
   (d) $4\,\mathbf{a} + 3\,\mathbf{b}$.
   (e) Draw a diagram showing the vectors $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j}$
   (f) On the diagram in (e) show the vector $\mathbf{a} + \mathbf{b}$

5. Place the vectors $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j}$ into
   (a) Column form
   (b) Geometric form.

6. A block of mass 30 kg is being dragged along a horizontal table by a force $\mathbf{F}$ of magnitude 15 N acting in the direction of 20^0 to the table.

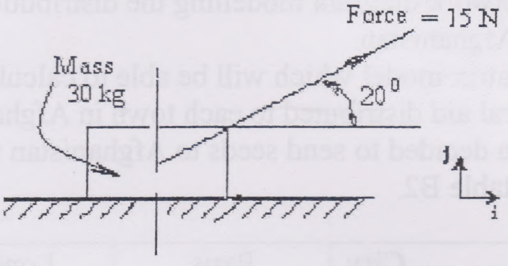
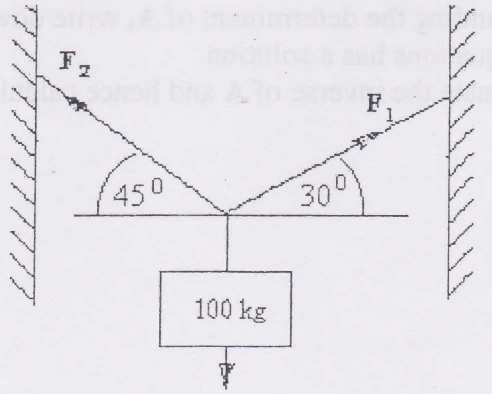


Fig. B1

   (a) Add the weight and Normal reaction force lines to the Fig B1 showing the direction of each force.
   (b) Model the pulling force, $\mathbf{F}$, by a vector in component form, stating the magnitude of the component which
       (i) accelerates the block along the table and
       (ii) acts in the same direction as the normal reaction to the table.
   (c) Write down the weight of the block as vector in component form and, using your result in (a)(ii), evaluate the Normal reaction to the table.

7. A load of mass 100kg is suspended by a light string system in static equilibrium, as shown in Fig. B2.
   (a) Write down the force equation connecting the tension in the strings and the weight of the load.
   (b) Draw the diagram showing the triangle of forces for this system.
   (c) By using the Sine Rule, evaluate the tensions in the strings giving the solutions in both component and geometric forms.

Fig. B2



8. Use

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

to evaluate (a) $\displaystyle\sum_{r=1}^{10} r$ (b) $\displaystyle\sum_{r=4}^{10} r$ (c) $\displaystyle\sum_{r=1}^{10} (r+1)$

9. The seal population of a particular North Sea island was numbered as 25 in the year 1990 and 175 in 2000.
   (a) Assuming the exponential model for population holds over the intervening period; evaluate the annual growth rate, r, to two significant figures.
   (b) Write down the population size in terms of the number of years, n, since 1990.
   (c) If this model continues to hold after 2000, what is the predicted seal population in 2020?
   (d) When will the population reach over 20 000?

10. (a) Establish whether the following sequences converge, and if so, find the limiting value.

(i) $a_n = 4 - (0.1)n \quad (n = 0,1,2,3 \ldots)$

(ii) $b_n = n/(2n+1) \quad (n = 0,1,2,3 \ldots)$

(iii) $c_n = \dfrac{14n+7}{7n+1} \quad (n = 0,1,2,3 \ldots)$

(b) A sequence x_n is generated by the recurrence relation

$$x_{n+1} = 0.4\left(x_n + \frac{4}{x_n}\right)$$

Establish algebraically, the limit to which this particular sequence may converge.

11. The population of seals at Rocket Mount each successive year is given by table B3.

1921	1922	1923	1924	1925	1926	1927	1928	1929	1930
4	16	63	250	1002	2019	4050	4100	4110	4115

Table B3

Assume that a logistic recurrence is to be used to model the population growth.

(a) Estimate a suitable value for r, assuming that the growth of the population between 1921 and 1924 is geometric.

(b) Estimate a value for E from the data. How confident are you about the accuracy of this estimate?

12. The system shown in Fig.B3 is in static equilibrium.

(a) Draw two diagrams showing the forces acting upon each mass.
(b) Assuming that the unit vector **i** acts up and parallel to the plane and the unit vector **j** acts normal to the plane, write down for the 40 kg mass the force equation.
(c) Determine the magnitude of the tension in the string and hence evaluate the mass M assuming $g = 9.81$ m s^{-2}.

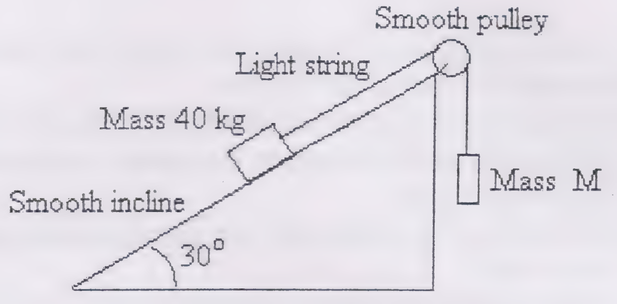


Fig. B3

MST121 Block C

Important ideas

- Differentiation is a process that enables you to find the
  gradient of a graph at any point on the graph
  rate at which one variable changes with respect to another.
- A derivative is defined in terms of a limit.
- An indefinite integral is a family of functions, an integral is a function, and a definite integral is a number.
- Integration can be thought of as
  the reverse of differentiation
  the limit of a sequence of sums (Fundamental Theorem of Calculus).
- A definite integral may be used to find areas under curves.
- Modelling is the process of choosing and using an appropriate function in order to describe a real-life situation.

Important techniques

- Differentiating functions (using tables and the constant multiple and sum rules).
- Optimisation.
- Integrating by using 'reverse differentiation'.
- Integration by guesswork and tables.
- Curve sketching.
- Finding general and particular solutions to differential equations.
- Evaluating definite integrals and relating the result to an area under a curve.
- Using a summing process to write a definite integral.
- Modelling using linear, quadratic, exponential, logarithmic and oscillating functions.

MST121 Block C Problems

1. Differentiate the following functions, identifying any general results that you use.

   (i) $f(x) = x^3 + \sqrt{x}$

   (ii) $g(y) = y^4 - \dfrac{1}{y^2} \qquad (y \neq 0)$

   (iii) $h(x) = 3x^2 + 2x - 1$

   (iv) $f(x) = (2x+3)(x-4)$

   (v) $g(t) = e^{4t} - \ln(2t)$

   (vi) $h(y) = 3\sin(2y) + \cos(4y) - e^{3y}$

2. Find the derivatives of the following, using the Product, Quotient or Composite Rules;

   (i) $y = \cos(3x)\ln(3x)$

   (ii) $s = \dfrac{e^{5t}}{2t^2 + 3}$

   (iii) $m = \sin\left(3n^2 - \dfrac{1}{n^2}\right)$

3. The distance s metres from a point O after t seconds is given by the function

   $$s(t) = t^3 + 7t + 12$$

   Find

   (i) the distance travelled after 3 seconds and after 4 seconds. Hence find the distance travelled in the fourth second.

   (ii) the velocity after 3 seconds,

   (iii) the acceleration after 4 seconds.

4. (a) Find the stationary points of the function

   $$f(x) = 4x^2(3-x).$$

   (b) Classify each of the stationary points,

   (i) using the First Derivative Test,

   (ii) using the Second Derivative Test.

   (iii) Find the y-coordinates of each of the stationary points on the graph and the x-coordinates of the points where the graph cuts the x-axis. Hence draw a rough sketch of the graph of this function.

5. Find the indefinite integrals of the following functions.

(i) $f(t) = 3t^2 + \sqrt{t}$

(ii) $g(x) = \cos(2\pi x) - 2\sin x$

(iii) $h(y) = \dfrac{2}{y} + e^{3y}$ $(y > 0)$

(iv) $f(x) = (x-3)(2x+1)$

(v) $g(x) = \dfrac{\sqrt{x} - 3}{x^2}$ $(x > 0)$

(vi) $h(w) = e^{2w}\left(e^w + 1\right)$

(vii) $f(x) = (3 - \sin x)(1 + \cos x)$

(viii) $f(x) = x^3 \sin\left(x^4 - 4\right)$

(ix) $g(x) = \dfrac{2x^2 + 1}{4x^3 + 6x}$ $(x > 0)$

(x) $h(x) = \cot x$ Note $\cot(x) = \dfrac{\cos x}{\sin x}$

6. A car travelling at 20 ms^{-1} accelerates uniformly so that in the next two seconds it covers 42 metres. What is its acceleration and its final speed?

7. Lois Lane is standing on top of the Eiffel Tower, 300m above the ground, when she slips off. With origin at the top of the tower, an axis vertically down and assuming the acceleration due to gravity is 10 ms^{-2}, the equation describing her acceleration is given by:

$$\frac{dv}{dt} = 10$$

(a) Find an expression for the velocity at which Lois will be falling after time t.

(b) Find an expression for the distance that Lois will have fallen after time t.

(c) How long will it take Lois to reach the ground?

Superman hears her cry and immediately flies to her rescue. He arrives at the top of the Eiffel Tower 2 seconds later and dives vertically downward, with an initial velocity of 10 ms^{-1} to try to catch her. He can dive with a super-acceleration that is twice that of the acceleration due to gravity.

(d) Find an expression for the velocity at which Superman will be travelling t seconds after diving off the top of the Tower.
(e) Find an expression for the distance that he will have dived after time t.
(f) Show that Superman will catch Lois.

8. Evaluate the following integrals.

(i) $\int_2^3 (x^2 - 5x + 4)dx$

(ii) $\int_{\pi/4}^{\pi/2} \sin(4y)dy$

9. Find the area under the graph of $y = e^{x/2} + 2$ from $x = 1$ to $x = 4$.

10. Find the general solutions of the differential equations:

(i) $\dfrac{dy}{dx} = 2x^2 + e^{3x}$

(ii) $\dfrac{1}{y^3}\dfrac{dy}{du} = 2\sin u - 1$

11. (a) Find the general solution of the differential equation

$$\frac{dy}{dx} = 5x^2 e^{-y}$$

(b) Find the particular solution of this equation that satisfies the initial condition $y = 1$ when $x = 0$.

12. Solve in explicit form the initial-value problem

$$\frac{dy}{dx} = y^2 \cos 3x, \text{ when } x = \frac{\pi}{2}, \text{ then } y = 1$$

13. The size S of a population at time t satisfies approximately the differential equation

$$\frac{dS}{dt} = kS$$

where k is a constant.

(i) Find the general solution to this equation.

(ii) The population numbered 32 000 in the year 1900 and had increased to 48 000 by 1970. Estimate what its size will in the year 2004. (Give your answer to the nearest 1000.)

14. On very steep hills where vehicles may go out of control (e.g. after brake failure) there are often lanes full of sand. One model of a large vehicle entering such a lane gives a differential equation involving the vehicle's velocity v ms^{-1} at time t s after entering the sand as $\frac{dv}{dt} = -0.2v$.

(a) Find

(i) the general solution of this differential equation.

(ii) the particular solution for a vehicle entering at 20 ms^{-1} and hence the time taken to decrease to a speed of 1 ms^{-1}.
Theoretically, how long would it take to come to a standstill?

(b) By integrating the solution of (a)(ii), find an expression for the distance s m travelled in the sand in time t. How long does the escape lane need to be so that the vehicle can stop safely?

15. A tank contains a salt solution (salt dissolved in water), At time t s the salt concentration in the solution is $c(t)$ g/l. Water containing a salt concentration of 3g/l runs into the tank at a constant rate and solution flows out of the tank at the same rate. The solution in the tank is kept perfectly mixed, so that the concentration of salt within it is uniform.
The change in concentration can be modelled by the differential equation

$$c'(t) = -(c(t) - 3)/200, \qquad c(t) > 3.$$

(a) Find

(i) the general solution of this differential equation.

(ii) the particular solution of the equation in part (i) for which the concentration in the tank at time $t = 0$ is 7g/l?

(b) To what limiting value does $c(t)$ tend as t becomes large? Explain why this value is what would be expected from physical considerations.

(c) Determine how long it takes for the concentration of salt to be within 5 % of the equilibrium level.

MST121 Block D

Important ideas

- There are two basic approaches to probability:
  theoretical (depends on the idea of equally likely outcomes)
  empirical (depends on experimentation).
- Uncertainty in a random variable can be represented by a probability distribution.
- Computers assist statistical exploration by:
  simulation (to investigate probability models)
  easy calculation and presentation of statistical information.
- The normal (Gaussian) distribution is widely applicable in modelling variation.
- Sample statistics may be used to make inferences about the populations from which the samples were drawn.
- The sampling distribution of the mean:
  changes its shape and reduces its standard deviation with increasing sample size
  may be approximated by a normal distribution (Central Limit Theorem).
- Boxplots may be used to make visual comparisons between samples of data.
- The difference between sample means can support inferences about the difference between the means of the populations from which the samples were drawn.
- Straight lines may be good models for the relationships between linked variables.
- The least squares fit line is the line having the lowest sum of squared residuals.

Important techniques

- Calculating probabilities using the basic laws of probability.
- Calculating the mean and standard deviation of a sample of data.
- Using sample mean, $\overline{x}$, and sample standard deviation, s, to make estimates of the corresponding population parameters.
- Using the model of a normal distribution to answer questions about a population.
- Constructing a 95% confidence interval for a population mean.
- Using the Central Limit Theorem to make inferences about a population.
- Using boxplots to compare two sets of data.
- Using the two-sample z-test (to a specified level of significance) to test a hypothesis about two population means.
- Obtaining the least-squares fit line (regression line) for a set of data, and using it for prediction.

MST121 Block D Problems

1. An experimenter carries out the following experiment to select a pair of cards from a pack:

   shuffle the pack;
   select a card and write down its suit;
   return the card to the pack;
   repeat for a second card.

a) What is the probability that the experimenter selects a pair of hearts?

b) The experiment is repeated six times (so that six pairs of cards are selected).

   i) Find the probability that no pair of hearts is selected.
   ii) Find the probability that there is at least one selection that produces a pair of hearts.
   iii) How many times would you expect the experimenter to have to repeat the experiment in order to obtain a pair of hearts? (That is, what is the mean number of repetitions required?)
   iv) How many times would you expect the experimenter to have to repeat the experiment in order to have a greater than 50% chance of obtaining a pair of hearts?

2.a) Twelve MST121 students are each asked to choose at random a number between 1 and 100 inclusive. Show that there is about an evens chance that two or more of them will pick the same number.

b) If twelve of the student group present at the Weekend were asked to take part in an experiment to test out the result in part a), what do you think would be the practical difficulties?

3. A sample of twelve children in a class each estimate the height of the top of a church tower. Their estimates, in metres, are:

47, 52, 52, 54, 52, 50, 51, 50, 48, 53, 54, 49.

Find the sample mean and sample standard deviation of their estimates.

4. A researcher records pulse rates for a random sample of 57 males and 35 females. The summary statistics are given below (measured in beats per minute).

	Males	Females
Sample Size	57	35
Mean	70.42	76.86
Standard deviation	9.948	11.62

The researcher proposes to use the two sample z-test to investigate whether or not there is a difference between the population mean pulse rate for males and the population mean pulse rate for females.

a) Write down appropriate null and alternate hypotheses, explaining the meanings of any symbols that you use.

b) Calculate the test statistic. Complete the test at the 5% level of significance. What conclusions should the researcher draw?

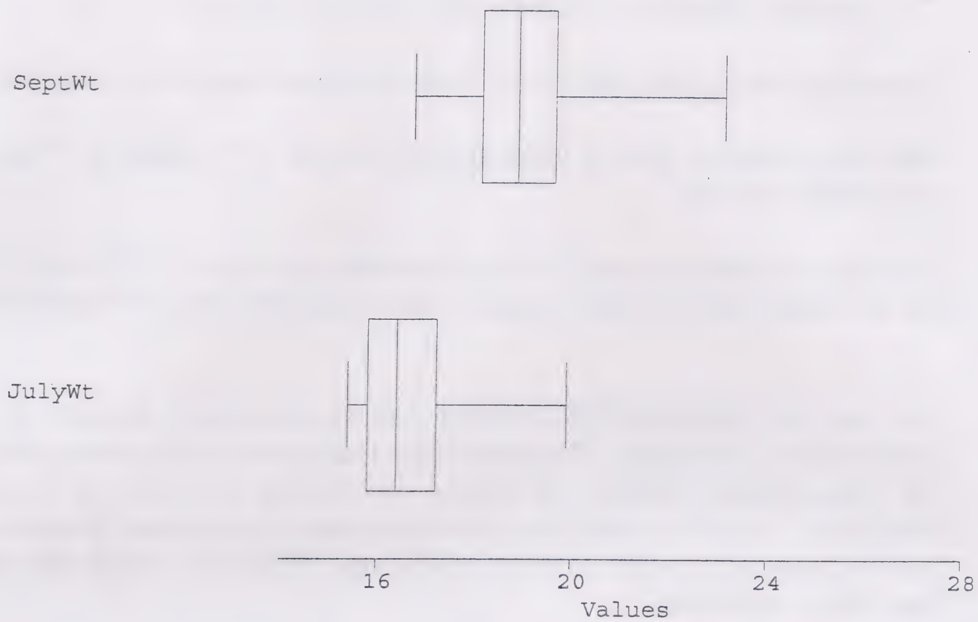
5. Packages from a packing machine have masses which should be normally distributed with mean 200g and standard deviation 3.2g.

a) Within what range of values should the masses of 95% of the packages lie?

b) A package is selected at random and it is found to have a mass of 193g. Would you consider that it was abnormally light?

c) A quality control inspector suspects that the operating company is deliberately producing underweight goods. She randomly samples sixteen of the packages, and computes their mean mass. Within what range of values will 95% of such means lie? Explain the difference between this answer and that to part a). What would the quality control inspector feel about a mean mass of 193 g?

6. The boxplots shown below summarise the data for the weights (in grams) of two samples of blackcaps, one obtained in September 1994, just prior to migration, and the other obtained in July 1994. The plots are followed by the corresponding summary statistics:



Summary statistics for:	SeptWt	JulyWt
Mean $\bar{x}$	18.94	16.72
Standard deviation, s	1.38	1.074
Minimum	16.7	15.4
Lower quartile, $Q1$	18.1	15.8
Median	18.85	16.4
Upper quartile, $Q3$	19.6	17.2
Maximum	23.1	19.9
Sample size, n	30	31

a) Mark on each boxplot the values of the median, the lower and upper quartiles, the maximum and the minimum. What do the boxplots tell you about the relative weights of the blackcaps in July and September?

b) Calculate the range and the interquartile range of the weights in the two different months and comment on your answers.

7. In an attempt to increase yield (measured in kg) of an industrial process, a technician varies the percentage of a certain additive used. All other conditions are kept as constant as possible. Data are collected showing the yield obtained, y, for percentage additive, a. These appear to be linearly related so that a least-squares fit line can be used as a model of the relationship.

The equation of this line is found to be $y = 1.17a + 127.15$

a) According to the model what is the mean yield from an additive of 4.2%?

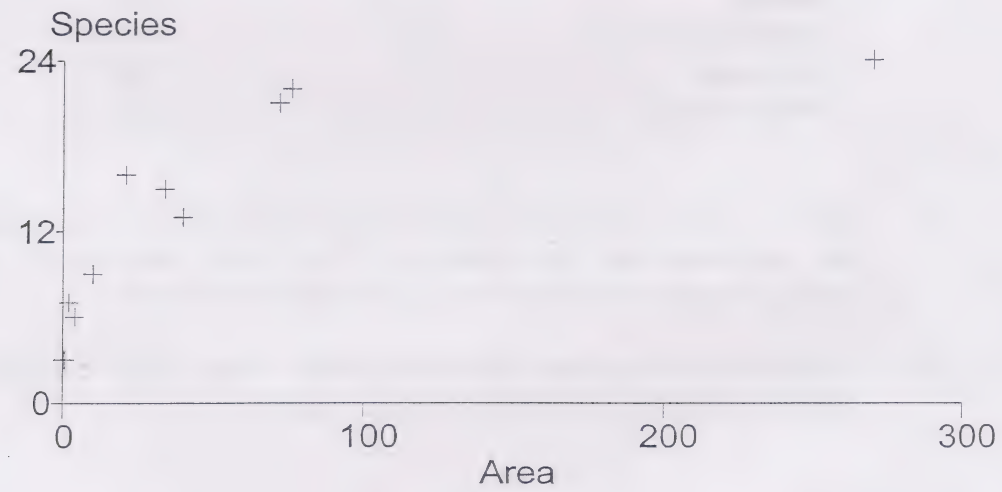
b) One pair of results gives a yield of 133.3 kg for 5.5% additive. What is the residual for this pair?

c) The range of additives used in this experiment was from 2.5% to 6%. Should the technician use this model to predict the mean yield from 10% additive?

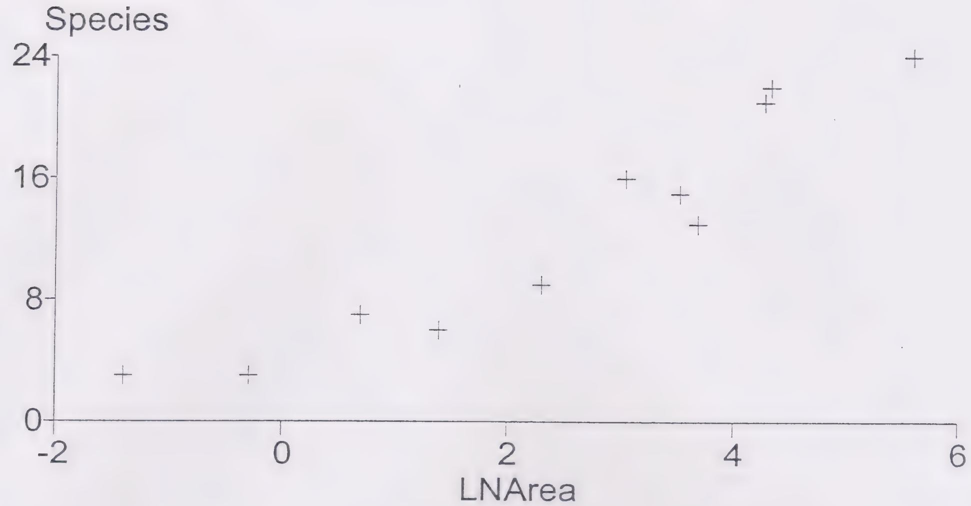
8. The data file ISLAND.OUS contains data on the area in hectares of eleven small islands in Shetland. The areas of the islands are in the column AREA and the corresponding number of species of breeding birds are in the column SPECIES. The four scatterplots on the next page show various attempts to find the best linear relationship between AREA and SPECIES. Which plot gives the best linear relationship?

Using your preferred plot, draw in what you think is the line of best fit by eye. Obtain approximate values for the intercept and gradient of the line you have drawn, and thus express the relationship (approximately) between AREA and SPECIES.

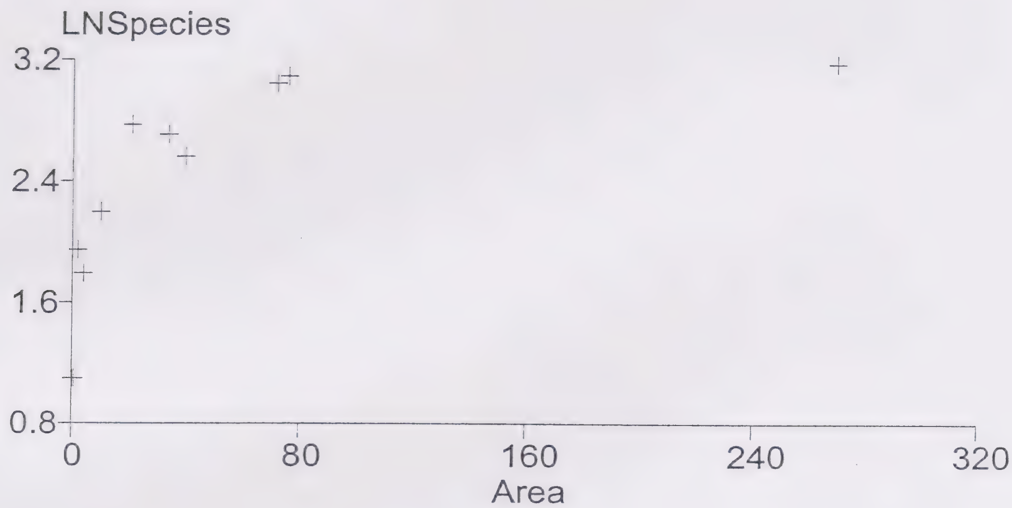
i)



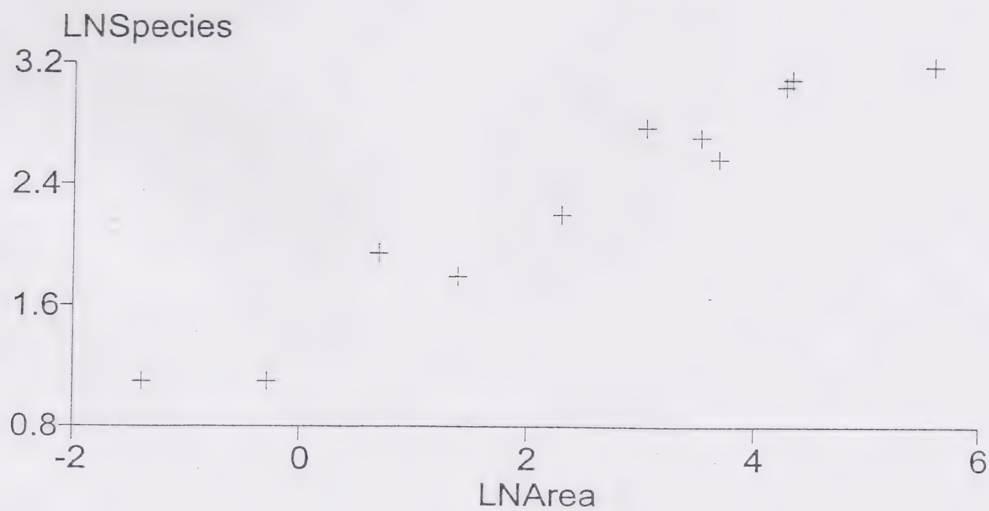
ii)


Species
24
16
8
0
-2
0
2
4
6
LNArea

iii)


LNSpecies
3.2
2.4
1.6
0.8
0
80
160
240
320
Area

iv)


LNSpecies
3.2
2.4
1.6
0.8
-2
0
2
4
6
LNArea

SOLUTIONS

MST 121 Block A Solutions

1. (i) $2(x + 2) - 3(x - 3) = 0$

   so $2x + 4 - 3x + 9 = 0$

   $-x = -13$

   and $\underline{x = 13}$

   (ii) $2x - 3y = 7$ ie $y = (2x - 7)/3$

   $3x + 5y = 1$ ie $y = (1 - 3x)/5$

   so $(2x - 7)/3 = (1 - 3x)/5$

   $5(2x - 7) = 3(1 - 3x)$

   $10x - 35 = 3 - 9x$

   $19x = 38$

   $\underline{x = 2}$

   then $y = (2 \times 2 - 7)/3$ ie $\underline{y = -1}$

   OR $2x - 3y = 7$ so $6x - 9y = 21$

   $3x + 5y = 1$ so $6x + 10y = 2$

   $-19y = 19$

   $\underline{y = -1}$

   so $2x + 3 = 7$

   $\underline{x = 2}$

2. (i) (a) $a = 1, b = 6, c = 9$ giving $b^2 - 4ac = 36 - 36 = 0$ so there is 1 solution.

   (b) $x^2 + 6x + 9 = 0$ or $(x + 3)^2 = 0$ giving $\underline{x = -3}$

   Formula $x = (-6 \pm \sqrt{(6^2 - 4\times1\times9)})/2 = -6/2 = \underline{-3}$

   (ii) (a) $a = 2, b = 1, c = -3$ giving $b^2 - 4ac = 1 + 24 > 0$ so there are 2 solutions.

   (b) $2x^2 + x - 3 = 0$ or $(2x + 3)(x - 1) = 0$ giving $\underline{x = -3/2 \text{ or } x = 1}$

   Formula $x = (-1 \pm \sqrt{(1^2 - 4\times2\times-3)})/4 = (-1 \pm 5)/4 = \underline{1 \text{ or } -3/2}$

   (iii) (a) $a = 1, b = 0, c = -9$ giving $b^2 - 4ac = 0 + 36 > 0$ so there are 2 solutions.

   (b) $x^2 - 9 = 0$ or $(x - 3)(x + 3) = 0$ giving $\underline{x = 3 \text{ or } x = -3}$

   Formula $x = (0 \pm \sqrt{(0 - 4\times1\times-9)})/2 = \pm 6/2 = \underline{3 \text{ or } -3}$

   (iv) (a) $b^2 - 4ac = 0 - 36 < 0$ so there are no real solutions.

3. (i) (a) $x_1 = 4000$

   $x_2 = 4000 \times 0.8 = 3200$

   $x_3 = 3200 \times 0.8 = 2560$

   $x_4 = 2560 \times 0.8 = 2048$

   (b) Geometric, $a = 4000, r = 0.8$

   $x_n = 4000 \times (0.8)^{n-1}$ $(n = 1, 2, 3\ldots)$

   (c) $x_1 = 4000 \times (0.8)^0 = 4000$

   $x_2 = 4000 \times (0.8)^1 = 3200$

   $x_3 = 4000 \times (0.8)^2 = 2560$

   $x_4 = 4000 \times (0.8)^3 = 2048$ as in (a)

   (d) $x_{10} = 4000 \times (0.8)^9 = 536.87$

   (ii) (a) $x_1 = -350$

   $x_2 = -350 - 50 = -400$

   $x_3 = -400 - 50 = -450$

   $x_4 = -400 - 50 = -500$

   (b) Arithmetic, $a = -350, d = -50$

   $x_n = -350 + (n - 1)(-50)$

   $x_n = -300 - 50n$ $(n = 1, 2, 3\ldots)$

   (c) $x_1 = -300 - 50 = -350$

   $x_2 = -300 - 50 \times 2 = -400$

   $x_3 = -300 - 50 \times 3 = -450$

   $x_4 = -300 - 50 \times 4 = -500$ as in (a)

   (d) $x_{10} = -300 - 50 \times 10 = -800$

   (iii) (a) $x_0 = 1024$

   $x_1 = -0.5 \times 1024 = -512$

   $x_2 = -0.5 \times -512 = 256$

   $x_3 = -0.5 \times 256 = -128$

   (b) Geometric, $a = 1024, r = -0.5$

   $x_n = 1024(-0.5)^n$ $(n = 0, 1, 2\ldots)$

   (c) $x_0 = 1024(-0.5)^0 = 1024$

   $x_1 = 1024(-0.5)^1 = -512$

   $x_2 = 1024(-0.5)^2 = 256$

   $x_3 = 1024(-0.5)^3 = -128$ as in (a)

   $x_9 = 1024(-0.5)^9 = -2$

4. (i) (a) $x_0 = 1$

$x_1 = 0.5 + 3 = 3.5$

$x_2 = 0.5 \times 3.5 + 3 = 4.75$

$x_3 = 0.5 \times 4.75 + 3 = 5.375$

Closed form

$a = 1,\ d = 3,\ r = 0.5$

$$x_n = \left(1 + \frac{3}{0.5 - 1}\right)\cdot(0.5)^n - \frac{3}{0.5 - 1}$$

$= -5\,(0.5)^n + 6 \quad (n = 0, 1, 2\ldots)$

Long term behaviour

As $n \to \infty,\ x_n \to 6 \quad ((0.5)^n \to 0)$

(b) Graph (iv)

(ii) (a) $x_0 = 1$

$x_1 = 2 + 3 = 5$

$x_2 = 2 \times 5 + 3 = 13$

$x_3 = 2 \times 13 + 3 = 29$

Closed form

$a = 1,\ d = 3,\ r = 2$

$$x_n = \left(1 + \frac{3}{2 - 1}\right)\cdot(2)^n - \frac{3}{2 - 1}$$

$= 4 \times 2^n - 3 \quad (n = 0, 1, 2\ldots)$

Long term behaviour

As $n \to \infty,\ x_n \to \infty \quad (2^n \to \infty)$

(b) Graph (iii)

(iii) (a) $x_0 = 1$

$x_1 = -0.5 + 3 = 2.5$

$x_2 = -0.5 \times 2.5 + 3 = 1.75$

$x_3 = -0.5 \times 1.75 + 3 = 2.125$

Closed form

$a = 1,\ d = 3,\ r = -0.5$

$$x_n = \left(1 + \frac{3}{-0.5 - 1}\right)\cdot(-0.5)^n - \frac{3}{-0.5 - 1}$$

$= -1(-0.5)^n + 2 \quad (n = 0, 1, 2\ldots)$

Long term behaviour

As $n \to \infty,\ x_n \to 2 \quad ((-0.5)^n \to 0)$

(oscillating)

(b) Graph (i)

(iv) (a) $x_0 = 1$

$x_1 = -2 + 3 = 1$

$x_2 = -2 \times 1 + 3 = 1$

$x_3 = -2 \times 1 + 3 = 1$

Closed form

$a = 1,\ d = 3,\ r = -2$

$$x_n = \left(1 + \frac{3}{-2 - 1}\right)\cdot(-2)^n - \frac{3}{-2 - 1}$$

$= 1 \quad (n = 0, 1, 2\ldots)$

Long term behaviour

$x_n = 1$ (constant)

(b) Graph (ii)

5. x_n (a) $3.5 = r + d$

$7.25 = 3.5r + d$

so $3.75 = 2.5r$

giving $\underline{r = 1.5}$

hence $\underline{d = 2}$

Recurrence system

$x_1 = 1, \quad x_{n+1} = 1.5x_n + 2 \quad (n = 1, 2, 3\ldots)$

(b) Closed form

$$x_n = \left(1 + \frac{2}{1.5 - 1}\right)\cdot(1.5)^{n-1} - \frac{2}{1.5 - 1}$$

$= 5(1.5)^{n-1} - 4 \quad (n = 1, 2, 3\ldots)$

(c) Long term behaviour

As $n \to \infty,\ x_n \to \infty$

y_n (a) $-16 = 20r + d$

$9.2 = -16r + d$

so $-25.2 = 36r$

giving $\underline{r = -0.7}$

so $-16 = -14 + d$

hence $\underline{d = -2}$

Recurrence system

$y_1 = 20, \quad y_{n+1} = -0.7y_n - 2 \quad (n = 1, 2, 3\ldots)$

(b) Closed form

$$y_n = \left(20 - \frac{2}{-0.7 - 1}\right)\cdot(-0.7)^{n-1} + \frac{2}{-0.7 - 1}$$

$= 21.1765(-0.7)^{n-1} - 1.1765 \quad (n = 1, 2, 3\ldots)$

(c) Long term behaviour

As $n \to \infty,\ x_n \to -1.1765$

(oscillating)

6. (a) (i) $AB = \sqrt{(2+2)^2 + (2+1)^2}$
$= \sqrt{16+9}$
$\underline{AB = 5}$

(ii) Gradient of AB = 3/4
so equation of AB is
$y + 1 = 3(x+2)/4$
ie $\underline{y = 3x/4 + 1/2}$

.(b) (i) Equation is $y = 14/3 - 4x/3$
so gradient = -4/3
and gradient of AB = ¾
Now $-4/3 \times ¾ = -1$
so the lines are perpendicular.

(ii) When x = 2, $y = 14/3 - 8/3 = 2$
so line passes through B

(c) See solution to Question 1(i). So point of intersection is (2, -1).

(d) (i) Gradients are 3 and 3 so lines are parallel.
(ii) Gradients are 3 and –3 so lines are neither parallel nor perpendicular.
(iii) Gradients are 3 and –1/3 and $3 \times -1/3 = -1$ so lines are perpendicular.

7. (i) Circle (a) (2, -3) (b) $\sqrt{7}$ (c) $x = 2 + \sqrt{7}\cos\vartheta$, $y = -3 + \sqrt{7}\sin\vartheta$.
(ii) Not circle (squares should be added).
(iii) Not circle (squares should have same coefficient).
(iv) Not circle (r^2 is negative).
(v) Using $z^2 + 2pz = (z+p)^2 - p^2$
$$x^2 + 4x + y^2 - 6y + 4 = 0$$
becomes $(x+2)^2 - 4 + (y-3)^2 - 9 + 4 = 0$
or $(x+2)^2 + (y-3)^2 = 9$
Therefore this is a circle with
(a) (-2, 3) (b) 3 (c) $x = -2 + 3\cos\vartheta$, $y = 3 + 3\sin\vartheta$.
(vi) Comparing with (v) this equation becomes
$$(x+2)^2 + (y-3)^2 = -1$$
which is not a circle as r^2 is negative.
(vii) $y^2 - 2x + x^2 - 5 = 0$
or $x^2 - 2x + y^2 - 5 = 0$
becomes $(x-1)^2 - 1 + (y-0)^2 - 5 = 0$
or $(x-1)^2 + (y-0)^2 = 6$
Therefore this is a circle with (a) (1, 0) (b) $\sqrt{6}$ (c) $x = 1 + \sqrt{6}\cos\vartheta$, $y = \sqrt{6}\sin\vartheta$
(viii) Not a circle ($-y^2$).
(ix) Not a circle (no y^2).

8. <u>Perpendicular bisector of AB.</u>
Gradient of AB = (2 + 2)/(3 – 1) = 2 so gradient of perpendicular = - ½.
Midpoint of AB = ((3 + 1)/2, (-2 + 2)/2) = (2, 0)
So equation of perpendicular bisector of AB is $y - 0 = -(x-2)/2$ or $\underline{y = -x/2 + 1}$
<u>Perpendicular bisector of AC.</u>
Gradient of AC = (1 + 2)/(-2 –1) = -1 so gradient of perpendicular = 1
Midpoint of AC = ((1 – 2)/2, (-2 + 1)/2) = (-1/2, -1/2)
So equation of perpendicular bisector of AC is $y + ½ = x + ½$ or $\underline{y = x}$

These lines intersect when $-x/2 + 1 = x$ ie $x = 2/3$ and $y = 2/3$.
So centre of circle, O, is at <u>(2/3, 2/3)</u>
Distance OA = $\sqrt{(1-2/3)^2 + (-2-2/3)^2} = \sqrt{1/9 + 64/9} = \underline{\sqrt{65/9}}$

Therefore the equation of the circle is $\underline{(x - 2/3)^2 + (y - 2/3)^2 = 65/9}$

9. (i) $AC/12 = \cos 65°$ so $AC = 12\cos 65° = \underline{5.07\text{cm (2 dps)}}$

$AB/12 = \sin 65°$ so $AB = 12\sin 65° = \underline{10.88\text{cm (2 dps)}}$

$\angle B = 90 - 65 = \underline{25°}$

(ii) $\dfrac{2.9}{DE} = \sin(40^o)$ so $\dfrac{2.9}{\sin(40^o)} = DE$ DE=4.51 cm 2 dp

$\dfrac{2.9}{DF} = \tan(40^o)$ so $\dfrac{2.9}{\tan(40^o)} = DF$ DF=3.46 cm 2 dp

$\angle E = 180^o - 90^o - 40^o = 50^o.$

(iii)By Pythagoras $20^2 = 15^2 + HI^2$ so $HI^2 = 20^2 - 15^2$ and $\underline{HI = 13.23\text{cm (2 dps)}}$

$\cos G = 15/20$ so $\angle G = \underline{41.4° \text{ (1dp)}}$

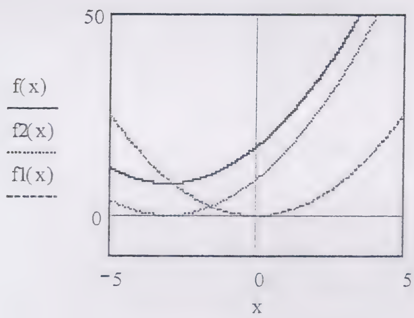
$\angle I = 90 - 41.4 = \underline{48.6° \text{ (1dp)}}$

10.(i) $f(x) = [x^2 + 6x] + 17$
$= [\{x + (6/2)\}^2 - (6/2)^2] + 17$
$= [\{x + 3\}^2 - 3^2] + 17$
$= (x + 3)^2 + 8$

A horizontal translation 3 units left;
A vertical translation 8 units up.

$f1(x) := x^2 \quad f2(x) := (x+3)^2$

$f(x) := f2(x) + 8$



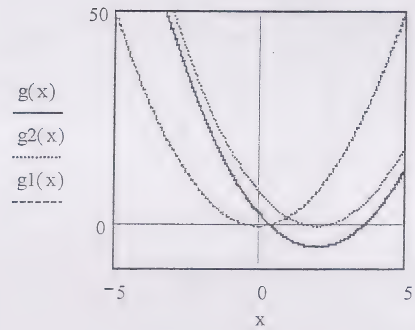
y-intercept $f(0) = 17$
x-intercept none; $f(x) = 0$ no sol'ns.

(ii) $g(x) = [2x^2 - 8x] + 3$
$= 2\,[x^2 - 4x] + 3$
$= 2\,[(x - 2)^2 - (-2)^2] + 3$
$= 2\,(x - 2)^2 - 2\,(-2)^2 + 3$
$= 2\,(x - 2)^2 - 5.$

A y-scaling with factor 2;
A horizontal translation 2 units right;
A vertical translation 5 units down.

$g1(x) := 2\cdot(x)^2 \quad g2(x) := 2\cdot(x-2)^2$

$g(x) := g2(x) - 5$

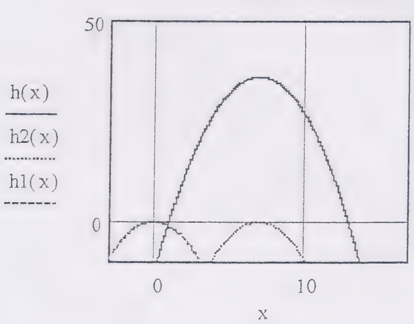


y-intercept $g(0) = 3$
x-intercept $g(x) = 0$ so $x \approx 0.42$ or 3.58
using formula for quadratic equation

(iii) $h(x) = 14x - 13 - x^2$
$= -x^2 + 14x - 13$
$= -[x^2 - 14x] - 13$
$= -[(x - 7)^2 - 7^2] - 13$
$= -(x - 7)^2 + 7^2 - 13$
$= -(x - 7)^2 + 36.$

A y-scaling with factor -1;
A horizontal translation 7 units right;
A vertical translation 36 units up.
y-intercept $h(0) = -13$
x-intercepts $h(x) = (x - 1)(13 - x) = 0$
$x = 1$ or 13.

$h1(x) := -x^2 \quad h2(x) := -(x-7)^2$
$h(x) := h2(x) + 36$

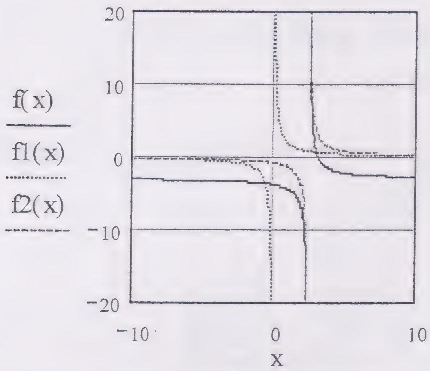


11 (i)

$$f(x)=\frac{4}{2\cdot x-5}-3=\frac{2}{x-2.5}-3$$

$$f1(x):=\frac{2}{x} \quad f2(x):=\frac{2}{x-2.5} \quad f(x):=f2(x)-3$$

A y-scaling with factor 2; a horizontal translation 2.5 units right; a vertical translation 3 units down



y-intercept $f(0) = 4/(-5) -3 = -3.8$

x-intercept $f(x) = 0$

$$\frac{4}{2\cdot x-5}-3=0$$

$4 - 3(2x - 5) = 0$

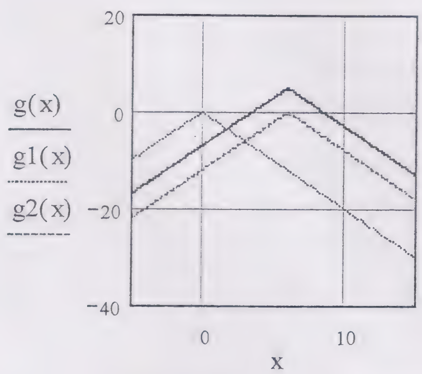
$x = 19/6$

(ii) $g(x) = -2\,|x - 6| + 5$

$$g1(x):=-2\cdot|x| \quad g2(x):=-2\cdot|x-6|$$

$$g(x):=g2(x)+5$$

A y-scaling with factor -2; a horizontal transation 6 units right; vertical translation 5 units up.



y-intercept $g(0) = -2|-6| +5$

$= -12 +5$

$= -7$

x-intercepts $g(x) = 0$

$-2\,|x - 6| + 5 = 0$

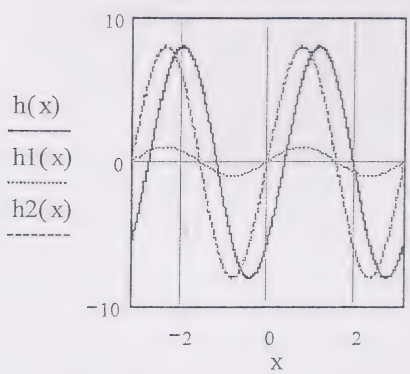
$|x - 6| = 5/2$

$x = 3.5$ or 8.5

(iii)

$$h(x)=8\cdot\sin\left(2\cdot x-\frac{\pi}{4}\right)=8\,\sin\left[2\cdot\left(x-\frac{\pi}{8}\right)\right]$$

$$h1(x):=\sin(2\cdot x) \qquad h2(x):=8\cdot\sin(2\cdot x) \qquad h(x):=h2\left(x-\frac{\pi}{8}\right)$$



An x-scaling with factor 1/2

A y-scaling with factor 8

A horizontal translation right of $\pi/8$ units

y – intercept

$$h(0)=8\cdot\sin\left(\frac{-\pi}{4}\right)=-4\cdot\sqrt{2}$$

x - intercepts

$$\sin\left(2\cdot x-\frac{\pi}{4}\right)=0 \qquad 2\cdot x-\frac{\pi}{4}=n\pi \qquad x=\frac{n\pi}{2}+\frac{\pi}{8} \qquad n=\ldots -2,-1,0,1,\ldots..$$

12. (a) (i) $5^x = 3125 = 5^5$

$x = 5$

(ii) $5^x = 0.0016 = 16/10000$

$= 1/625 = 5^{-4}$

$x = -4$

(b) (i) $7^x = 3125$

$\ln(7^x) = \ln(3125)$

$x \ln 7 = \ln(3125)$

$x = \ln(3125)/ \ln 7$

≈ 4.135

(ii) $e^{-2x} = 37$

$\ln(e^{-2x}) = \ln 37$

$-2x\ln(e) = \ln 37$

$-2x = \ln 37$

$x = (\ln 37)/(-2)$

≈ -1.805

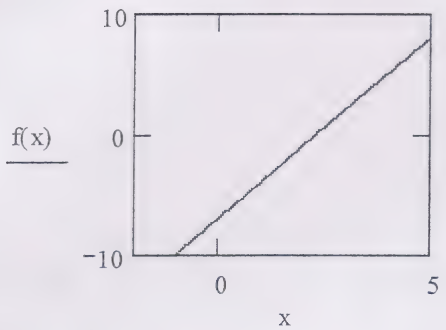
(c) (i) $\ln(x/3) = 17$

$x/3 = e^{17}$

$x = 3e^{17}$.

13(i)

(a) $f(x) := 3 \cdot x - 7.$



(b) Let $y = 3x - 7$

Then $3x = y + 7$

and $x = \dfrac{y+7}{3}$

Function f is one-one so has an inverse, domain R and image set R.

$$f^{-1}(x) = \frac{x+7}{3}$$

(ii) (a) see solution to 11(i) above

(b) Function g is one-one so does have an inverse. Domain R, $(x \neq -3)$.

Image set R, $(x \neq 2.5)$.

Let $y = \dfrac{4}{2x-5} - 3$

Then $y + 3 = \dfrac{4}{2x-5}$ and $(y + 3)(2x - 5) = 4$

$(y + 3).2x = 4 + 5(y + 3)$

$x = \dfrac{5y+19}{2(y+3)}$

Inverse function is $g^{-1}(x) = \dfrac{5x+19}{2(x+3)}$

(iii)(a) See solution to 10(ii) above.

(b) h does not have an inverse - it is not one-one.

14. (i) Let $y = \cos(2x)$

Then $\arccos y = 2x$

so $x = (\arccos y)/2$

and $f^{-1}(x) = (\arccos x)/2$

Domain $[-1,1]$ Image set $[0, \pi/2]$

(ii) Let $y = \tan(x/4)$

Then $\arctan y = x/4$

so $x = 4 \arctan y$

and $g^{-1}(x) = 4 \arctan x$

Domain $(-\infty, \infty)$ Image set $(-2\pi, 2\pi)$

(iii) Let $y = 4\ e^{-(3x)} + 7$

then $y - 7 = 4\ e^{-(3x)}$

$$e^{-3x} = \frac{y-7}{4}$$

$$-3x = \ln\left(\frac{y-7}{4}\right)$$

$$x = -\frac{1}{3} \ln\left(\frac{y-7}{4}\right)$$

Inverse function is given by

$$h^{-1}(x) = \frac{-1}{3} \ln\left(\frac{x-7}{4}\right)$$

Domain $(7, \infty)$, image set R

(iv) Let $y = 5 \ln(3x-2) - 8$

then $y + 8 = 5 \ln(3x--2)$

and $\ln(3x-2) = (y+8)/5$

$3x - 2 = \exp\{y+8)/5\}$

$$x = \frac{1}{3}[\exp\{y+8)/5\} + 2]$$

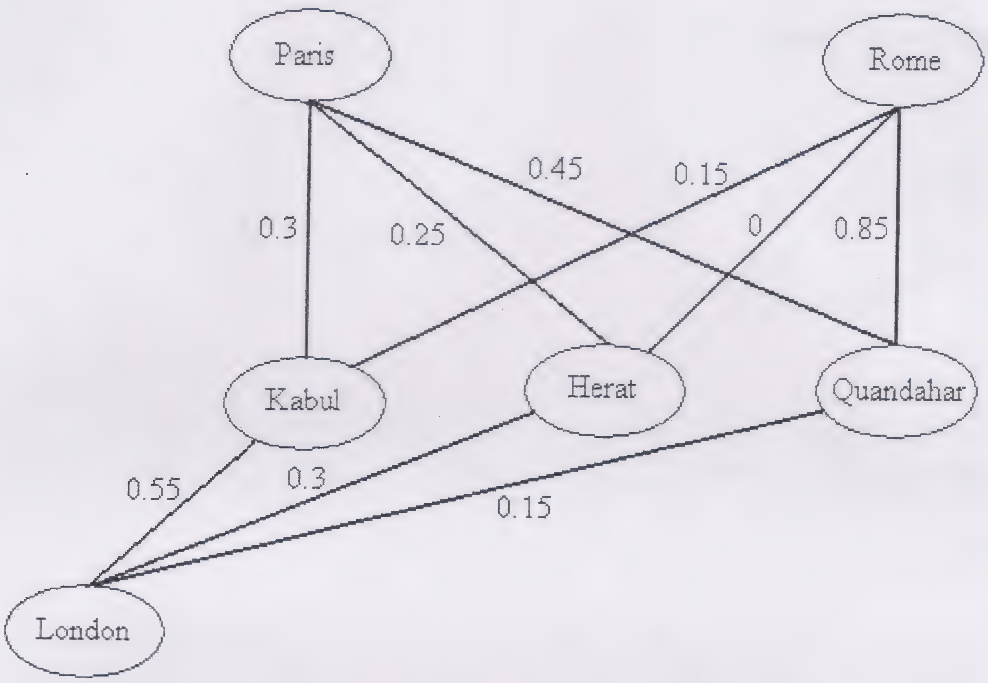
Inverse function is given by

$$k^{-1}(x) = \frac{1}{3}[\exp\{x+8)/5\} + 2]$$

Domain R, image set $(2/3, \infty)$

MST121 Block B Solutions

1. The network diagram shown may be drawn using a different layout for the nodes but showing the same connections and figures.



(b) $$\begin{pmatrix} 0.3 & 0.55 & 0.15 \\ 0.25 & 0.30 & 0.00 \\ 0.45 & 0.15 & 0.85 \end{pmatrix} \begin{pmatrix} P \\ L \\ R \end{pmatrix} = \begin{pmatrix} K \\ H \\ Q \end{pmatrix}$$

The letters within the vectors are the first letter of the town or city's name; for example, H stands for Herat.

(c) $$\begin{pmatrix} K \\ H \\ Q \end{pmatrix} = \begin{pmatrix} 1485 \\ 750 \\ 2465 \end{pmatrix}$$

Hence Kabul receives 1485 tonne , Herat, 750 tonne and Quandahar, 2465 tonne of seeds.

2. (a)

$$A + B = \begin{pmatrix} 0 & 4 \\ 1 & 7 \end{pmatrix} \qquad A - B = \begin{pmatrix} 2 & 0 \\ -7 & 1 \end{pmatrix}$$

$$A - 2 \cdot B = \begin{pmatrix} 3 & -2 \\ -11 & -2 \end{pmatrix} \qquad 3 \cdot A - 4 \cdot B = \begin{pmatrix} 7 & -2 \\ -25 & 0 \end{pmatrix}$$

det(A) = 10   det(B) = -11

(b)

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix}$$

(c)

$$B^{-1} = \frac{1}{-11}\begin{pmatrix} 3 & -2 \\ -4 & -1 \end{pmatrix}$$

(d)

$$A \cdot B = \begin{pmatrix} 7 & 8 \\ 19 & 6 \end{pmatrix} \qquad B \cdot A = \begin{pmatrix} -7 & 6 \\ -5 & 20 \end{pmatrix}$$

3.

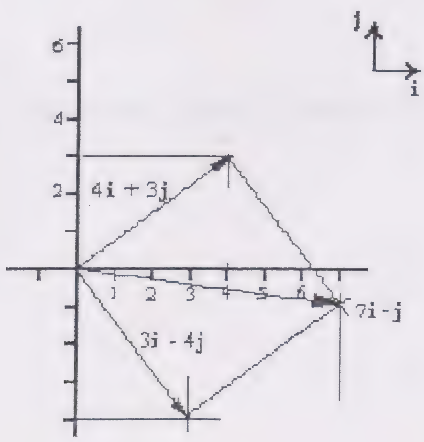
$$\mathbf{A}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathbf{b} \qquad \begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \end{pmatrix}$$

(a) The determinant of matrix A is –16. This is non-zero so the system of equations has a solution.

(b)

$$\mathbf{A}^{-1} = -\frac{1}{16}\begin{pmatrix} -2 & -4 \\ -3 & 2 \end{pmatrix} \qquad \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathbf{A}^{-1} \cdot \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

4. (a) **a** + **b** = 7 **i** - **j**
(b) **a** - **b** = - **i** - 7**j**
(c) 2**a** - 3**b** = -6 **i** - 17 **j**
(d) 4**a** + 3**b** = 24 **i** - 7 **j**
(e)



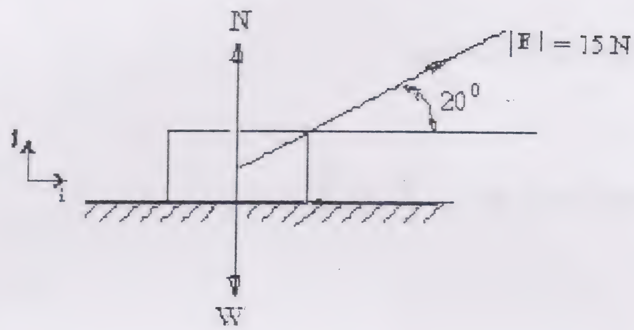
5.

$$\mathbf{a} := \begin{pmatrix} 3 \\ -4 \end{pmatrix} \qquad \mathbf{b} := \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Geometric forms $\quad |\mathbf{a}| = 5 \quad \theta = -51.13\,°$

$|\mathbf{b}| = 5 \quad \theta = 36.87\,°$

6. (a)



(b) $\mathbf{F} = 15\cos(20^0)\,\mathbf{i} + 15\sin(20^0)\,\mathbf{j}$

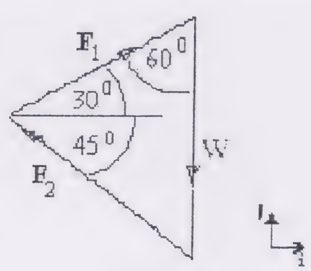
(i) Magnitude of the component of **F** which accelerates the block along the table is $15\cos(20^0)$ N.

(ii) Magnitude of the component of F which is in the same direction as the normal reaction to the table is $15\sin(20^0)$ N.

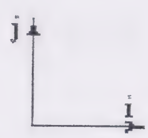
(iii) Weight of block $= -30g\,\mathbf{j}$ $(g = 9.81\ \text{ms}^{-2})$ and the normal Reaction from the table is $\mathbf{N} = (30g - 15\sin(20^0))\,\mathbf{j}$

7. (a) $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{W} = \mathbf{0}$ where $|\mathbf{W}| = 100g$ $(g = 9.81\ \text{ms}^{-2})$

(b) This is one of two possible diagrams. Can you sketch the other one?



(c) Define unit vectors **i** and **j** by diagram



Use the Sine Rule to obtain $|\mathbf{F}_1| = 718.14$ N, $\theta = 30^0$

$|\mathbf{F}_2| = 879.54$ N, $\theta = 135^0$

With unit vectors **i** and **j** we have $\mathbf{F}_1 = |\mathbf{F}_1|\cos(30^0)\,\mathbf{i} + |\mathbf{F}_1|\sin(30^0)\,\mathbf{j}$

$\mathbf{F}_2 = -|\mathbf{F}_2|\cos(45^0)\,\mathbf{i} + |\mathbf{F}_2|\sin(45^0)\,\mathbf{j}$

8. (a) 55
(b) 49
(c) 65

9. (a) $r = 0.214814$

(b) $P_{n+1} = 25(1+r)^n$

(c) Population after 30 years is 8575

(d) Number of years n is nearly 5 (4.932)

10. (a) (i) Limit does not exist
(ii) 0.5
(iii) 2

(c) Possible limits are $\pm\sqrt{\frac{8}{3}}$

11. (a) $P_n = (1+r)^n P_o$

$\Rightarrow \quad 250 = (1+r)^3 \times 4$

$\Rightarrow \quad (1+r)^3 = \frac{250}{4}$

$\Rightarrow \quad r = \left(\frac{250}{4}\right)^{1/3} - 1 = 2.96 \approx 3$

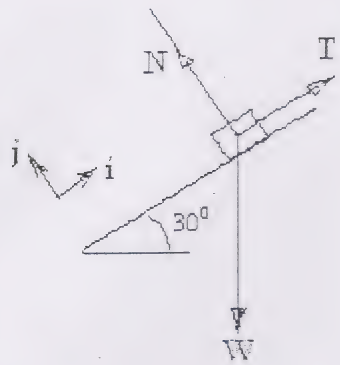
(b) $E \approx 4200$

Try modelling the population by

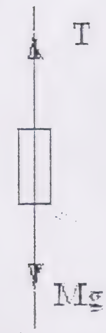
$$P_{n+1} - P_n = 3P_n\left(1 - \frac{P_n}{4200}\right)$$

then modify the constants 3 and 4200 to give a better fit to the data.

12. (a) For mass 40 kg



For mass M



(b) For mass M kg $\quad |\mathbf{T}| = Mg \quad$ (where g is the gravitational acceleration)

As system is in equilibrium, for the mass 40 kg we have,

$$|\mathbf{T}|\,\mathbf{i} - |\mathbf{W}| \sin(30^0)\,\mathbf{i} - |\mathbf{W}| \cos(30^0)\,\mathbf{j} + |\mathbf{N}|\,\mathbf{j} = \underline{\mathbf{0}}$$

$$|\mathbf{T}|\,\mathbf{i} - 40g \sin(30^0)\,\mathbf{i} - 40g \cos(30^0)\,\mathbf{j} + |\mathbf{N}|\,\mathbf{j} = \underline{\mathbf{0}}$$

Equating **i** components $|\mathbf{T}|\,\mathbf{i} - 40g \sin(30^0)\,\mathbf{i} = \underline{\mathbf{0}}$

or $\quad |\mathbf{T}| = 40g \sin(30^0) = 40g/2 = 20g$ N

Therefore $\quad Mg = 20g \quad$ or $\quad M = 20$ kg.

MST121 Block C Solutions

1.(i) $f'(x) = 3x^2 + \frac{1}{2}x^{-½}$ or $f'(x) = 3x^2 + \frac{1}{2\sqrt{x}}$, using the Sum Rule.

(ii) $g'(y) = 4y^3 - (-2y^{-3})$ or $g'(y) = 4y^3 + \frac{2}{y^3}$, using the Sum Rule.

(iii) $h'(x) = 6x + 2$, using Sum and Constant Multiple Rules.

(iv) First it is a good idea to multiply out the brackets to give

$f(x) = (2x+3)(x-4) = 2x^2 - 5x - 12$.
So, $f'(x) = 4x - 5$, using the Sum and Constant Multiple Rules.

(v) $g'(t) = 4e^{4t} - \frac{1}{t}$, using the Sum Rule.

(vi) $h'(y) = 6\cos(2y) - 4\sin(4y) - 3e^{3y}$, using the Sum and Constant Multiple Rules.

2.(i) $\frac{dy}{dx} = -3\sin(3x)\ln(3x) + \cos(3x) \times \left(\frac{1}{x}\right)$, using the Product Rule.

This is better written as $\frac{dy}{dx} = -3\sin(3x)\ln(3x) + \frac{1}{x}\cos(3x)$.

(iii) Using the Quotient Rule,

$$\frac{ds}{dt} = \frac{(2t^2+3)5e^{5t} - e^{5t}(4t)}{(2t^2+3)^2} \quad \text{or} \quad \frac{ds}{dt} = \frac{e^{5t}(10t^2 - 4t + 15)}{(2t^2+3)^2}$$

(iv) Using the Composite Rule,

$$\frac{dm}{dn} = \cos\left(3n^2 - \frac{1}{n^2}\right) \times \left(6n - (-2n^{-3})\right) \text{ or } \frac{dm}{dn} = \left(6n + \frac{2}{n^3}\right)\cos\left(3n^2 - \frac{1}{n^2}\right)$$

3.(i) The distance travelled after 3 seconds is given by
$s(3) = 3^3 + 7 \times 3 + 12 = 60$ m.

The distance travelled after 4 seconds is given by
$s(4) = 4^3 + 7 \times 4 + 12 = 104$ m.

So the distance travelled in the fourth second is 104 – 60 = 44m.

(ii) The velocity is given by $s'(t) = 3t^2 + 7$. After 3 seconds this is
$s'(3) = 3 \times 3^2 + 7 = 34$ ms^{-1}.

(iii) The acceleration is given by $s''(t) = 6t$. After 4 seconds this is $6 \times 4 = 24$ ms^{-2}.

4.(a) The derivative of the function $f(x) = 4x^2(3-x) = 12x^2 - 4x^3$ is
$f'(x) = 24x - 12x^2 = 12x(2-x)$

Solving the equation $f'(x) = 0$, we find that the stationary points of f are
$x = 0$ and $x = 2$.

(b)(i) To classify the stationary point at $x = 0$, choose test points $x_L = -1$ and $x_R = 1$
This gives $f'(x_L) = f'(-1) = 24(-1) - 12(-1)^2 = -36 < 0$ and
$$f'(x_R) = f'(1) = 24(1) - 12(1)^2 = 12 > 0$$
so f has a local minimum at $x = 0$, by the First Derivative Test.

To classify the stationary point at $x = 2$, choose test points $x_L = 1$ and $x_R = 3$
This gives $f'(x_L) = f'(1) > 0$ as seen above and
$$f'(x_R) = f'(3) = 24(3) - 12(3)^2 = -36 < 0$$
so f has a local maximum at $x = 2$, by the First Derivative Test.

(ii) The second derivative of f is $f''(x) = 24 - 24x$
For the stationary point at $x = 0$ $f''(0) = 24 > 0$
So f has a local minimum at $x = 0$, by the Second Derivative Test.

For the stationary point at $x = 2$ $f''(2) = -24 < 0$
So f has a local maximum at $x = 2$, by the Second Derivative Test.

(c) The y-coordinates of stationary points on the graph of $y = f(x)$ are
$f(0) = 0$ and $f(2) = 12(2)^2 - 4(2)^3 = 16$

Hence the graph passes through the points (0, 0), a local minimum and (2, 16), a local maximum.

$f(x) = 0$ if $x = 0$ or $x = 3$, so the graph also passes through the point (3,0).

5. (i) $\int(3t^2+\sqrt{t})dt = t^3+\frac{2}{3}t^{3/2}+c$

(ii) $\int(\cos(2\pi x)-2\sin x)dx = \frac{1}{2\pi}\sin(2\pi x)+2\cos x+c$

(iii) $\int\left(\frac{2}{y}+e^{3y}\right)dy = 2\ln y+\frac{1}{3}e^{3y}+c$

(iv) $\int(x-3)(2x+1)dx = \int(2x^2-5x-3)dx = \frac{2}{3}x^3-\frac{5}{2}x^2-3x+c$

(v) $\int\frac{\sqrt{x}-3}{x^2}dx = \int\left(\frac{\sqrt{x}}{x^2}-\frac{3}{x^2}\right)dx = \int\left(x^{-3/2}-3x^{-2}\right)dx = -2x^{-1/2}+3x^{-1}+c$

$$= -\frac{2}{\sqrt{x}}+\frac{3}{x}+c$$

(vi) $\int e^{2w}(e^w+1)dw = \int(e^{3w}+e^{2w})dw = \frac{1}{3}e^{3w}+\frac{1}{2}e^{2w}+c$

(vii) $\int(3-\sin x)(1+\cos x)dx = \int(3-\sin x+3\cos x-\sin x\cos x)dx$

Using $\sin(2\theta)=2\sin\theta\cos\theta$ then $\sin x\cos x = \frac{1}{2}\sin(2x)$

So the integral becomes

$$\int\left(3-\sin x+3\cos x-\frac{1}{2}\sin(2x)\right)dx = 3x+\cos x+3\sin x+\frac{1}{4}\cos(2x)+c$$

(viii) $\int x^3\sin(x^4-4)dx$

The factor x^3 in the integrand is, except for a constant multiple, the derivative of x^4-4. Making the necessary adjustment we get

$$\frac{1}{4}\int 4x^3\sin(x^4-4)dx = -\frac{1}{4}\cos(x^4-4)+c$$

(ix) $\int\frac{2x^2+1}{4x^3+6x}dx$

The numerator $2x^2+1$ in the intergrand is, except for a constant multiple, the derivative of the denominator $4x^3+6x$, so making the adjustment

$$\frac{1}{6}\int\frac{12x^2+6}{4x^3+6x}dx = \frac{1}{6}\ln(4x^3+6x)+c$$

(x) $\int\cot x\,dx = \int\frac{\cos x}{\sin x}dx$

Since the numerator is the derivative of the denominator, this gives

$$\int\cot x\,dx = \ln(\sin x)+c$$

6. If we take $t = 0$ and $s_0 = 0$ at the time when the car is travelling at 20 ms^{-1}, then from the information in the question we can extract
$v_0 = 20$, $t = 2$ $s_0 = 0$ $s = 42$ and we need to find a.

Using $s = \frac{1}{2}at^2 + v_0 t + s_0$ we get $42 = \frac{1}{2}a(2)^2 + 20(2) + 0$
So $a = 1$

Now using $v = at + v_0$ we get $v = 20 + 1\times2 = 22$
So the acceleration is 1 ms^{-2} and final speed is 22 ms^{-1}.

7.(a) The velocity v is given by $\int \frac{dv}{dt}dt = \int 10dt = 10t + c$
When $t = 0$ then $v = 0$, hence $c = 0$ so an expression for the velocity at which Lois falls is $v = 10t$

(b) The distance s is given by $\int vdt = \int 10tdt = 5t^2 + c$
When $t = 0$ then $s = 0$, hence $c = 0$ so an expression for the distance that Lois will fall is $s = 5t^2$

(c) As the Tower is 300m above then ground, substituting this in $s = 5t^2$ gives
$300 = 5t^2$ or $60 = t^2$
$t = \pm\sqrt{60} = \pm 7.7$ (to 1 dp).
Only the positive value is valid in the problem, so it will take Lois 7.7 seconds to reach the ground.

For Superman

(d) His acceleration is 20 ms^{-2} so $\frac{dv}{dt} = 20$
To find his velocity, v, we need $\int \frac{dv}{dt}dt = \int 20dt = 20t + c$
when $t = 0$, $v = 10$, hence $c = 10$ so an expression for the velocity at which Superman will be travelling t seconds after diving off the top of the Tower is $v = 20t + 10$

(e) The distance s is given by $\int vdt = \int (20t + 10)dt = 10t^2 + 10t + c$
When $t = 0$, $s = 0$, hence $c = 0$, giving the expression for the distance that he will have travelled as $s = 10t^2 + 10t$

(f) Substituting $s = 300$ into the above expression gives
$300 = 10t^2 + 10t$ this becomes
$t^2 + t - 30 = 0$ factorising
$(t - 5)(t + 6) = 0$ so $t = 5$ or $t = -6$
Only the positive value is valid so Superman will hit the ground 5 seconds after diving from the Tower. As he arrived at the top of the Tower 2 seconds after Lois fell, he will hit the ground 7 seconds after Lois falls, so Superman catches Lois.

8.(i) $\int_2^3 (x^2 - 5x + 4)dx = \left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x\right]_2^3$

$$= \left(\frac{1}{3}\times 3^3 - \frac{5}{2}\times 3^2 + 4\times 3\right) - \left(\frac{1}{3}\times 2^3 - \frac{5}{2}\times 2^2 + 4\times 2\right)$$

$$= \left(9 - \frac{45}{2} + 12\right) - \left(\frac{8}{3} - 10 + 8\right) = -\frac{13}{6}$$

(ii) $\int_{\pi/4}^{\pi/2} \sin(4y)dy = \left[-\frac{1}{4}\cos(4y)\right]_{\pi/4}^{\pi/2} = \left(-\frac{1}{4}\cos(2\pi)\right) - \left(-\frac{1}{4}\cos\pi\right)$

$$= -\frac{1}{4}(1) + \frac{1}{4}(-1) = -\frac{1}{2}$$

9. The graph of $y = e^{x/2} + 2$ is above the x axis since $e^{x/2} + 2$ is always positive, so the required area is

$$\int_1^4 \left(e^{x/2} + 2\right)dx = \left[2e^{x/2} + 2x\right]_1^4 = \left(2e^2 + 2\times 4\right) - \left(2e^{1/2} + 2\times 1\right) = 17.48 \text{ to 2 dp.}$$

10.(i) The general solution is $y = \int \left(2x^2 + e^{3x}\right)dx$

$$y = \frac{2}{3}x^3 + \frac{1}{3}e^{3x} + c$$

(ii) By integration we get $\int \frac{1}{y^3}dy = \int (2\sin u - 1)du$

$$-\frac{1}{2}y^{-2} = -2\cos u - u + c \qquad c \text{ an arbitrary constant}$$

Multiply through by –2 to give $\frac{1}{y^2} = 4\cos u + 2u + k$ where $k = 2c$

So general solution is $y = \dfrac{1}{\pm\sqrt{4\cos u + 2u + k}}$

11(a) This is of the correct form to use Separation of Variables to give

$$\frac{1}{e^{-y}}\frac{dy}{dx} = 5x^2 \quad \text{or} \quad e^y \frac{dy}{dx} = 5x^2$$

By integration with respect to x $\qquad \int e^y dy = \int 5x^2 dx$

$$e^y = \frac{5}{3}x^3 + c$$

Taking natural logarithms of each side gives general solution

$$y = \ln\left(\frac{5}{3}x^3 + c\right)$$

(b) When $x = 0$ and $y = 1$ then $\quad 1 = \ln(0 + c) = \ln c$ hence $c = e^1 = e$

So particular solution is $y = \ln\left(\frac{5}{3}x^3 + e\right)$

12. Separating variables and integrating, gives

$$\int \frac{1}{y^2} dy = \int \cos(3x) dx$$

$$-\frac{1}{y} = \frac{1}{3}\sin(3x) + c$$

$$y = \frac{-1}{\frac{1}{3}\sin(3x) + c}$$ This is the general solution

When $x = \frac{\pi}{2}$ then $\sin(3x) = -1$, also $y = 1$ (given), so

$1 = \frac{-1}{\frac{1}{3}(-1) + c}$, this leads to $c = -\frac{2}{3}$

Solution to the problem is $y = \dfrac{-1}{\frac{1}{3}\sin(3x) - \frac{2}{3}}$

13.(i) $\frac{dS}{dt} = kS$. $\qquad$ Separating to give $\frac{1}{S}\frac{dS}{dt} = k$, now integrating

$\int \frac{1}{S} dS = \int k dt$ this leads to $\ln S = kt + c$ where c is an arbitrary constant

Since $e^{\ln S} = S$ then $S = e^{(kt+c)} = e^{kt}e^{c}$
or $S = Ae^{kt}$ is the general solution (where $A = e^c$)

(ii) Take year 1900 as $t = 0$ and measuring S in thousands, then $S = 32$ when $t = 0$ so $A = 32$ and $S = 32e^{kt}$.

Also $S = 48$ when $t = 70$ (ie.1970) and so $48 = 32e^{70k}$ which implies that $70k = \ln(48/32)$ and thus $k \cong 0.0058$.

Finally, 2004 corresponds to $t = 104$ and $S = 32e^{104k} \cong 58.448$ and the population will be about 58 000.

14. (a) (i) Separating variables and integrating, gives

$$\int \frac{1}{v}\,dv = \int -0.2dt$$

$$\ln v = -0.2t + c$$

$$v = Ae^{-0.2t}$$

(ii) When t = 0 then v = 20, so

$20 = Ae^0$ which leads to A = 20 and $v = 20e^{-0.2t}$.

When v = 1 then $e^{-0.2t} = 0.05$, $e0^{.2t} = 20$, $0.2t = \ln 20$

so $t = 5 \ln 20 \cong 15$ s

Theoretically the vehicle comes to a standstill after an infinite time!

(b) From (a)(ii) $v = \dfrac{ds}{dt} = 20e^{-0.2t}$

Integrating $s = 20e^{-0.2t} / (-0.2) + c = -100e^{-0.2t} + c$

When $t = 0$ then $s = 0$, so $0 = -100 + c$ and $c = 100$

Thus $s = 100(1 - e^{-0.2t})$.

Thus for large values of t, s tends to 100 and so the escape lane must be at least 100 metres long.

15. (a) (i) Using Leibniz notation $\dfrac{dc}{dt} = c'(t)$ and $c = c(t)$

$$\frac{dc}{dt} = -(c-3)/200$$

$$\int \frac{1}{c-3}\,dc = \int -200dt$$

$$\ln(c - 3) = -200t + c$$

$$c - 3 = Ae^{-200t}$$

so the general solution is $c = 3 + Ae^{-200t}$

(ii) when t = 0, c =7 so 7 = 3 + A and A = 4

so particular solution is $c = 3 + 4e^{-200t}$.

(b) c tends to 3 as t becomes large, which is expected as the concentration becomes more diluted and eventually equals the incoming concentration.

(c) 5 % of 3 is 0.15, so we need to find time for c to reach 3.15. That is to find t such that

$$3.15 = 3 + 4e^{-200t}$$

$$e^{-200t} = 0.15/4 = 0.0375$$

$$200t = \ln(1/0.0375)$$

$$\text{so } t = 200 \ln(1/0.0375) \cong 657 \text{ s}$$

It takes approximately 10 minutes to reach within 5% of equilibrium level.

1 a) Probability of selecting one heart on a single draw $= \frac{13}{52} = \frac{1}{4}$.

Therefore, using the multiplication rule for independent events, the probability of selecting a pair of hearts $= \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$.

b) i) Probability that experiment does not result in a pair of hearts $= 1 - \frac{1}{16} = \frac{15}{16}$.

So probability that no pair of hearts is selected in six repetitions of experiment $= \left(\frac{15}{16}\right)^6 \approx 0.6789$.

ii) Using a), probability that at least one repetition produces a pair of hearts $=1 - 0.6789 = 0.3211$.

iii) The number of repetitions required to obtain a pair of hearts has geometric distribution with parameter $p = \frac{1}{16}$. The mean number of selections required to obtain a pair of hearts is therefore $\frac{1}{p} = \frac{1}{1/16} = 16$.

iv) We need to find the smallest n where $1 - \left(\frac{15}{16}\right)^n > 0.5$.

Consider the equation

$$1 - \left(\frac{15}{16}\right)^n = 0.5$$

Rearranging, this gives

$$0.5 = \left(\frac{15}{16}\right)^n$$

Taking natural logs gives

$$\ln(0.5) = n \ln\left(\frac{15}{16}\right)$$

$$n = \frac{\ln(0.5)}{\ln\left(\frac{15}{16}\right)}$$

$$\approx 11$$

2. a) Suppose that student A has chosen a number. Then the probability that student B's number is different is $\frac{99}{100}$. Then the probability that student C's number is different from both of the others is $\frac{98}{100}$. Continuing in this way, the probability that all twelve students have chosen different numbers is $\frac{99}{100} \times \frac{98}{100} \times \cdots \times \frac{89}{100} \approx 0.503$. Thus the probability that two or more have the same number is about one half, or an evens chance.

b) In practice, when people are asked to choose at random they do not do so (unless someone has provided them with a random number generator, and insists that it be used). Some numbers are more likely to be chosen than others - for example birth date numbers (i.e. numbers 1 - 31) have been chosen far more than any others for the National Lottery. The calculation done above assumes that all numbers are equally likely to be chosen.

3. The sample mean, $\bar{x} = \dfrac{47 + 52 + 52 + \ldots + 4}{12} = 51$ metres.

The sample standard deviation is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$$

$$= \sqrt{\frac{1}{11}\left((47-51)^2 + (52-51)^2 + (52-51)^2 + \ldots + (49-51)^2\right)}$$

$$= \sqrt{\frac{56}{11}} \approx 2.256304299 \approx 2.26 \text{ metres.}$$

4. a) $H_0 : \square_M = \square_F$

$H_1 : \square_M \neq \square_F$

where H_0 is the null hypothesis; H_1 is the alternative hypothesis; $\square_M$ is the mean pulse rate for the population of males and $\square_F$ is the mean pulse rate for the population of females.

b) The estimated standard error of the difference of the means is

$$ESE = \sqrt{\frac{s^2{}_M}{n_M} + \frac{s^2{}_F}{n_F}} = \sqrt{\frac{9.948^2}{57} + \frac{11.62^2}{35}} = 2.365169717\ldots$$

The test statistic is $\dfrac{\bar{x}_M - \bar{x}_F}{ESE} = \dfrac{70.42 - 76.86}{2.365169717\ldots} \approx -2.723.$

Since the test statistic z = –2.723 < –1.96 we reject the null hypothesis at the 5% level of significance.

The researcher should conclude that the mean pulse rate for the population of males is not equal to the mean pulse rate for the population of females.

The sample mean is higher for the females so this suggests that the mean pulse rate for the population of females is greater than the mean pulse rate for the population of males

5. a) According to the model, the masses of 95% of the packages will be within 1.96 standard deviations of the mean.

So the range is between 200 – 1.96 × 3.2 g and 200 + 1.96 × 3.2 g, which is approximately 193.7g to 206.3 g.

b) A value of 193g is not in this range, so it *would* seem to be rather light. Of course, 5% of the values are outside the range found in a), and half of these will be at the light end of the scale. Values below 193.7g occur 2.5% of the time - so the answer is down to what we mean by "abnormally light".

c) The standard error of the mean is $\dfrac{3.2}{\sqrt{16}} = \dfrac{3.2}{4} = 0.8$ g.

So the range is between 200 – 1.96 × 0.8 g and 200 + 1.96 × 0.8 g, which is approximately 198.4 g to 201.6g. This range is much narrower than that in part a). We could view part a) as illustrating samples of size $n = 1$; the increase to $n = 16$ leads to a scaling down of the standard deviation by factor of four. Thus the interval obtained here has width one quarter of that in a). The inspector would be very unhappy indeed about a mean of 193 g. The corresponding z-value is in fact equal to -8.75, which is virtually impossible if the machine is operating properly.

6. a) Many answers are possible here. Weights seem to be generally higher in September than July. Also the range of weights is greater in September (see part b)). Half (or more) of the July weights are below the lowest September weight. Three quarters (or more) of the July weights are below the lower quartile for September. Both sets of values are positively skewed (mean > median), with the skewness slightly more marked in the case of the July data (this has been confirmed by a calculation of the skewness using a more powerful statistical package which gave September skewness = 0.8159; July skewness = 1.0350). In both cases the most extreme high value is much further from the median than is the most extreme low value.

   Note that there can be difficulties in comparing two boxplots. It *can* be the case that all five of the statistics used in drawing two boxplots are higher in the first than the second, but that in fact the mean is lower in the second than the first. This is only possible when the data values are very unevenly spread across the two ranges. It is very unlikely indeed for a random sample from a normal distribution, and certainly not in our case since we also have the mean available to us. So it *is* reasonable for us to conclude here that weights seem to be generally higher in September than July.

   b) The range for September is 23.1 – 16.7 = 6.4 g.
   The range for July is 19.9 – 15.4 = 4.5 g. Thus the September values are more widely spread.
   The interquartile range for September is 19.6 –18.1 = 1.7 g.
   The interquartile range for July is 17.2 – 15.8 = 1.4 g. Thus the spread of the middle 50% is slightly greater in September but not by much.

7. a) Mean yield from a 4.2% additive is $1.17 \times 4.2 + 127.15 = 132.064$ kg.
   b) RESIDUAL = DATA - FIT
   FIT for 5.5% additive is $1.17 \times 5.5 + 127.15 = 133.585$ kg, so required RESIDUAL = 133.3 – 133.585 = –0.285 kg.
   c) The technician should not use this model for predictions for 10% additive. It is possible that the relationship remains linear for larger values of the additive but we have absolutely no evidence to support this. 10% is a lot higher than 6%, which was the highest value used to build the model. It is entirely possible that beyond a certain point performance actually decreases as a result of increasing the additive. A well-known example where this happens is in the application of fertiliser to crops. Using too much fertiliser actually damages the crops and they do not develop. For a wide range of lower values, however, there is an approximately linear relationship in which increased fertiliser leads to increased mean yield.

8. Plot iv) seems to be the best. The y-intercept is about 1.5 (you need to remember that the y-intercept corresponds to $x = 0$!). The gradient is about 0.35. So the equation is approximately LNSpecies = 1.5 + 0.35LNArea.

This equation can be transformed as follows:

$$\begin{aligned} \text{Species} &= \exp(1.5 + 0.35\text{LNArea}) \\ &= \exp(1.5) \times \text{Area}^{0.35} \\ &\approx 4.5\ \text{Area}^{0.35} \end{aligned}$$